Review Set for the course
"Discrete Mathematics" (2024)

Exercise 1. How many permutations of the set $[n]$ exist with the following conditions?

(1) Such that 1 and n are adjacent (i.e. next to each other).
(2) Such that 1 and n are separated by exactly one element.

Exercise 2. Prove the following identity for $n \geq 1$.

$$\sum_{A \subseteq [n]} \sum_{B \subseteq [n]} |A \cap B| = n4^{n-1}$$

Exercise 3. Let q_n be the number of nonconsecutive permutations of $[n]$, i.e. permutations of the set $[n]$ such that no consecutive numbers appear consecutively. For example, for $n = 4$, 1324 is nonconsecutive but 1243 is not.
Show that

$$q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$

Exercise 4. (a) Determine the generating function of triangular numbers $a_n = \binom{n}{2}$.

(b) What is the generating function of the sequence $(a_0, a_1, a_2 \ldots)$ with $a_0 = 1, a_1 = 3$ and $a_k = 3a_{k-1} - 2a_{k-2}$ for $k \geq 2$? Furthermore find an expression for the value of a_k.
(c) Find an expression for the value of b_k and the generating function, given that $b_0 = 0, b_1 = 1, b_2 = 2$ and $b_2, b_3, \ldots$ follow the recurrence given by

$$b_{k+3} = 5b_{k+2} - 7b_{k+1} + 3b_k.$$

Exercise 5. If F_n is the $n-$th Fibonacci number, prove that

$$F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

Exercise 6. Let m_n be the number of ways the vector $w = (n, 0)$ can be written as a sum

$$w = v_1 + v_2 + \cdots + v_n,$$

where each $v_i \in \{(1,0), (1,1), (1,-1)\}$ and for any $k \leq n$, the y-coordinate of $v_1 + v_2 + \cdots + v_k$ is non-negative. Such arrangements are called Motzkin paths. Can you see how similar or different they are to Dyck paths?

Set $m_0 = 1$. Find m_1, m_2, m_3, m_4. If $m(x) = \sum_{i=0}^{\infty} m_i x^i$ is their generating function, show that

$$x^2 m(x)^2 + (x-1)m(x) + 1 = 0$$

Exercise 7. Let $m \leq n$. Let $c_{m,n}$ be the number of ways in which n coins of 1 franc and m coins of 2 franc can be distributed among $m+n$ people standing in a coffee machine queue in the following way. Each person gets exactly one coin, and when they start buying a coffee in the order they are standing, the coffee machine never runs out of change. The coffee costs 1 franc and the machine has no coins inside to begin with.

Show that $c_{m,n} = \frac{n-m+1}{n+1}\binom{m+n}{n}$.

Exercise 8. For $n, k \in \mathbb{Z}_{\geq 1}$, consider the expression

$$\sum_{d|n} \mu(d) \log^k d.$$

Show that it is equal to 0 if n has more than k distinct prime divisors.

Exercise 9. Let G be a graph on 9 vertices. Suppose that the total sum of degrees in G is at least 27.

(a) Show that there is a vertex with degree at least 4.
(b) Does such a graph always have a triangle?

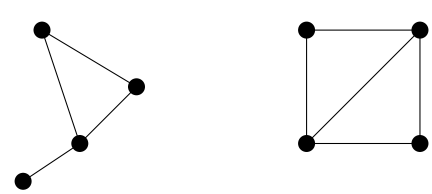
Exercise 10. Let G be a weighted, simple graph. Describe an algorithm which yields a *maximal* spanning tree on G.

Exercise 11. The purpose to this exercise is to give an alternative proof of Cayley's theorem.

(a) Compute the eigenvalues of the adjacency matrix of K_n;
(b) Show that λ is an eigenvalue of the Laplacian matrix of K_n if, and only if, $n-1-\lambda$ is an eigenvalue of the adjacency matrix of K_n;
(c) Use Kirchhoff's theorem to conclude Cayley's theorem.

Exercise 12. For G_1, G_2 two finite, simple graphs, we define the *Ramsey number of* G_1 *versus* G_2 – $r(G_1; G_2)$ – as the least integer n such that, in every 2-edge colouring of K_n, there is either a blue G_1 or a red G_2.

(a) Show that $r(C_4; C_4) \leq 6$.
(b) Show that $r(K_3; C_4) \leq 7$.
(c) Show that the item above can be strengthened as follows: if G_1 denotes a triangle with an appended extra vertex to one of its vertices, and G_2 is obtained by removing an edge from a K_4, show that $r(G_1; G_2) \leq 7$.



Graphs G_1 and G_2 as above.

Exercise 13. Six football teams participate in a round robin tournament. Any two teams play each other exactly once. We say that three teams beat each other if in their games played against each other, each team got one victory and one loss. What is the expected number of triples of teams who beat each other? Assume that each game is a toss-up, that is, each team has 50 percent chance to win any of its games.

Exercise 14. Prove that the expected number of cycles of a permutation of length n is

$$\sum_{k=1}^{n} \frac{1}{k}.$$